Chapter P: Preparation for Calculus

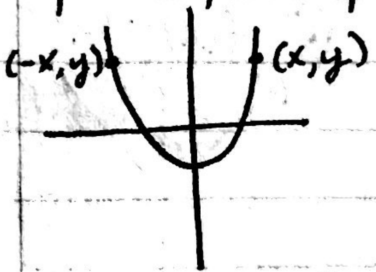
Solving for x and y intercepts

y intercept: set x to 0

x intercept: set y to 0

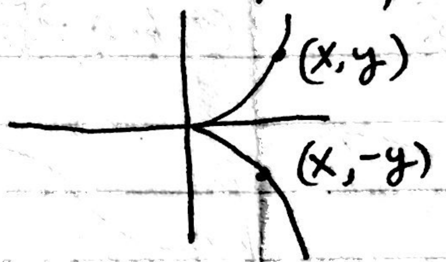
Testing for Symmetry

① y-axis symmetry



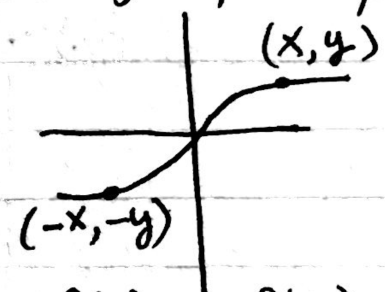
$f(x) = f(-x)$

② x-axis symmetry



$f(x) = -f(x)$

③ origin symmetry



$f(x) = -f(-x)$

The Slope of a Line

$m = \frac{\Delta y}{\Delta x}$

point slope form: $y - y_1 = m(x - x_1)$

slope intercept form: $y = mx + b$

general form: $ax + bx + c = 0$

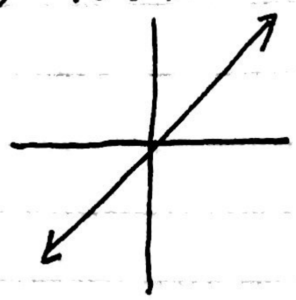
Parallel & Perpendicular Lines

if $m_1 = m_2 \Rightarrow$ parallel

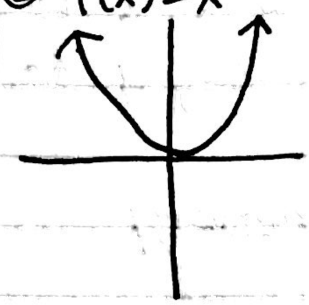
if $m_1 = -\frac{1}{m_2} \Rightarrow$ perpendicular

Parent Functions

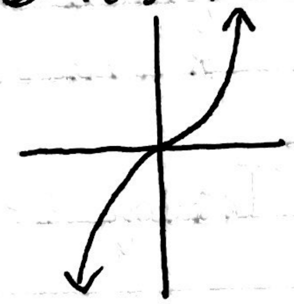
① $f(x)=x$



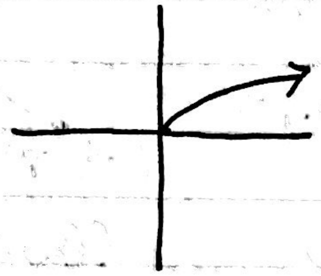
② $f(x)=x^2$



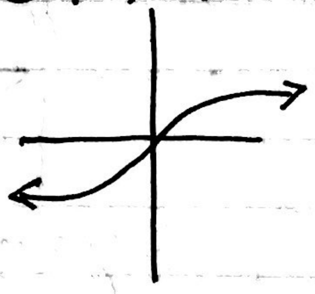
③ $f(x)=x^3$



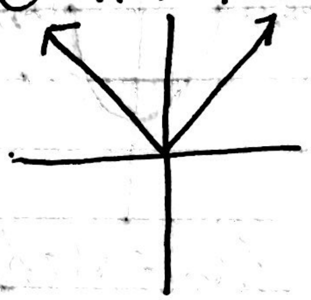
④ $f(x)=\sqrt{x}$



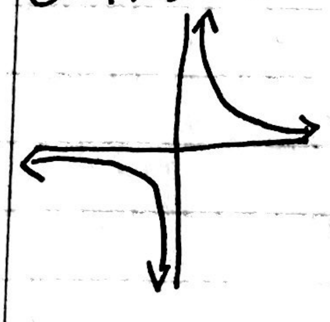
⑤ $f(x)=\sqrt[3]{x}$



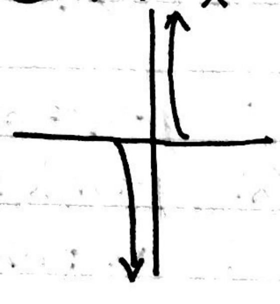
⑥ $f(x)=|x|$



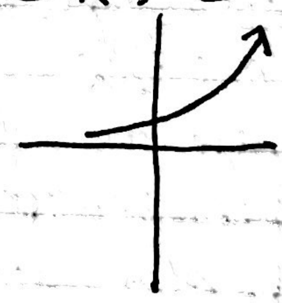
⑦ $f(x)=\frac{1}{x}$



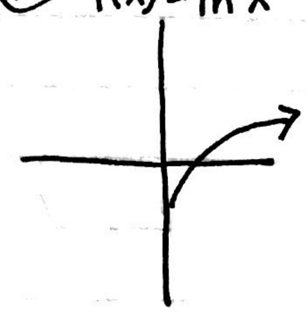
⑧ $f(x)=\frac{1}{x^3}$



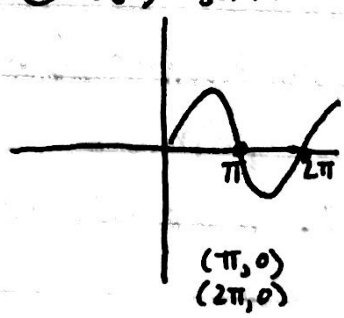
⑨ $f(x)=e^x$



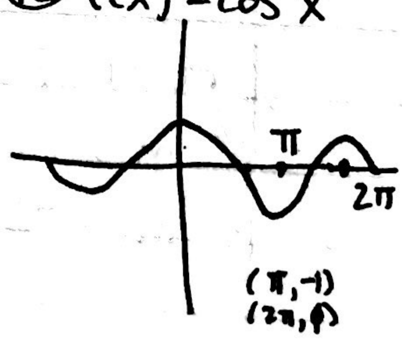
⑩ $f(x)=\ln x$



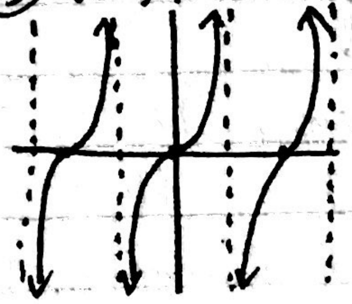
⑪ $f(x)=\sin x$



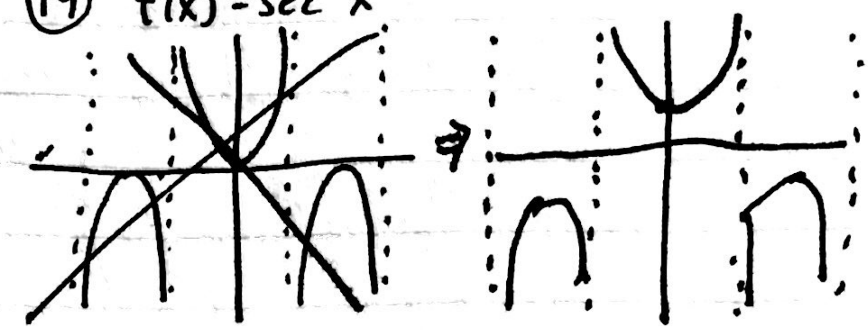
⑫ $f(x)=\cos x$



⑬ $f(x)=\tan x$



⑭ $f(x)=\sec x$



Important Trig Identities

reciprocal

$\sin u=\frac{1}{\csc u}$ $\cos u=\frac{1}{\sec u}$ $\tan u=\frac{1}{\cot u}$

$\csc u=\frac{1}{\sin u}$ $\sec u=\frac{1}{\cos u}$ $\cot u=\frac{1}{\tan u}$

Pythagorean

$\sin^2 u+\cos^2 u=1$ $1+\tan^2 u=\sec^2 u$ $1+\cot^2 u=\csc^2 u$

Quotient

$\tan u=\frac{\sin u}{\cos u}$ $\cot u=\frac{\cos u}{\sin u}$

Even-odd identities

$\sin(-u)=-\sin u$ $\cos(-u)=\cos u$ $\tan(-u)=-\tan u$

$\csc(-u)=-\csc u$ $\sec(-u)=\sec u$ $\cot(-u)=-\cot u$

Sum/difference

$\sin(u\pm v)=\sin u\cos v\pm\cos u\sin v$

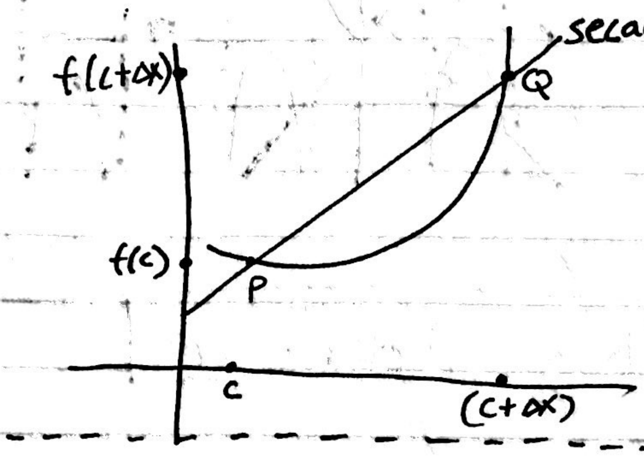
$\cos(u\pm v)=\cos u\cos v\mp\sin u\sin v$ *

$\tan(u\pm v)=\frac{\tan u\pm\tan v}{1\mp\tan u\tan v}$

* $\mp$

Chapter 1: Limits and their properties

The Tangent Line Problem



$P = (c, f(c))$

$Q = ((c+\Delta x), (f(c+\Delta x))$

therefore:

$$M_{sec} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{f(\cancel{x}+\Delta x) - f(c)}{(c+\Delta x) - c}$$

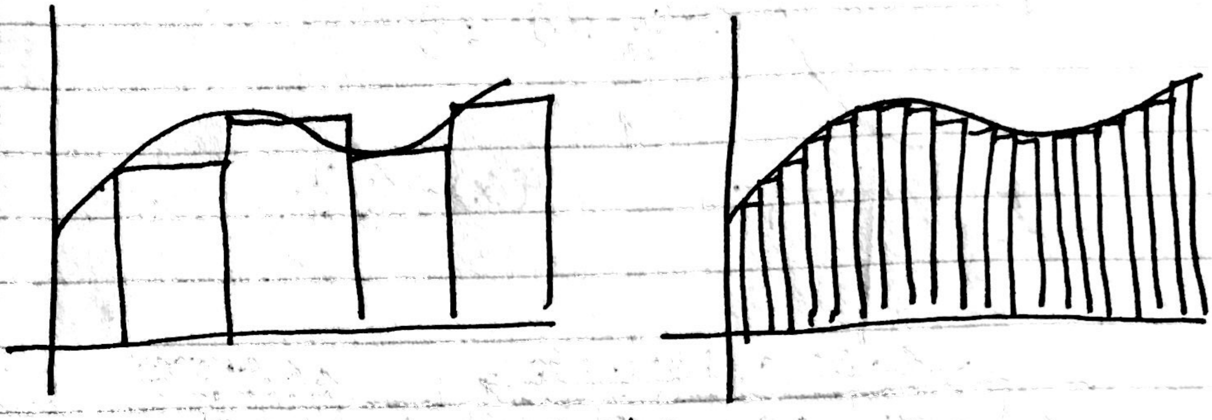
$$= \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

thus:

- The slope of a tangent line is the limit as Δx approaches 0 at the secant line

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

The Area Problem

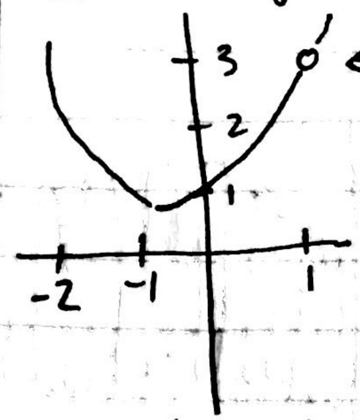


More rectangles = More accurate
(area estimation)

goal: determine limit of sum of areas of rectangle as number of rectangles increase without bound

These are the two fundamental problems in Calculus!!!

estimating a limit



$\leftarrow \lim_{x \to 1} f(x) = 3$

$$f(x) = \frac{x^3 - 1}{x - 1}$$

x approaches 1 from left → ← x approaches 1 from right

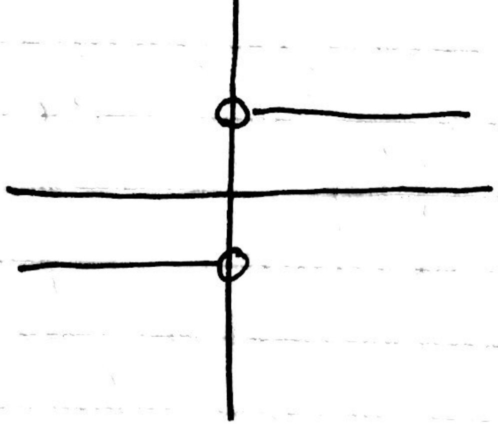
x	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25
$f(x)$	2.313	2.710	2.970	2.997	?	3.003	3.030	3.310	3.813

f(x) approaches 3 → ← f(x) approaches 3

$$\boxed{\lim_{x \to c} f(x) = L}$$

Limits that Fail to Exist

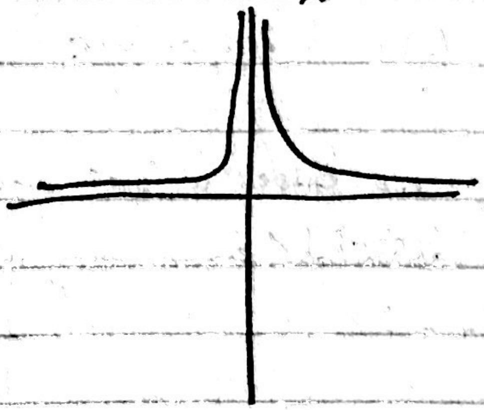
① $f(x) = \frac{|x|}{x}$ Behavior that differs from left and right



f(x) approaches different values from left and right

② Unbounded Behavior

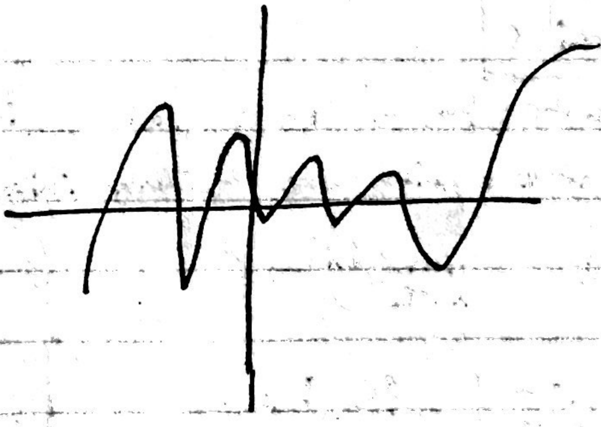
$f(x) = \frac{1}{x^2}$



asymptote at "0"

③ Oscillating Behavior

$f(x) = \sin \frac{1}{x}$



For all above examples, $\lim_{x \to 0} f(x) = DNE$

Definition of Limit

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number.

$$\lim_{x \to c} f(x) = L$$

means that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if

$0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Some Basic Limits

Let b and c be real numbers and let n be a positive integer

① $\lim_{x \to c} b = b$ ② $\lim_{x \to c} x = c$ ③ $\lim_{x \to c} x^n = c^n$

eg. $\lim_{x \to 2} 3 = 3$ $\lim_{x \to -4} x = -4$ $\lim_{x \to 2} x^2 = 2^2 = 4$

Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. Scalar multiple: $\lim_{x \to c} [b f(x)] = bL$

2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$

3. Product : $\lim_{x \to c} [f(x) g(x)] = LK$

4. Quotient $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}$ $K \neq 0$

5. Power $\lim_{x \to c} [f(x)]^n = L^n$

The Limit of a Composite Function

If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L)$$

e.g.

if $\left[\begin{array}{l} \lim_{x \to 0} (x^2+4) = 0^2+4 = 4 \\ \lim_{x \to 4} \sqrt{x} = \sqrt{4} = 2 \end{array}\right.$

then $\left[\ \text{limit}_{x \to 0} \sqrt{x^2+4} = \sqrt{4} = 2\right.$

Limits of Trigonometric Functions

① $\lim_{x \to c} \sin x = \sin c$

② $\lim_{x \to c} \cos x = \cos c$

③ $\lim_{x \to c} \tan x = \tan c$

④ $\lim_{x \to c} \cot x = \cot c$

⑤ $\lim_{x \to c} \sec x = \sec c$

⑥ $\lim_{x \to c} \csc x = \csc c$

e.g.

① $\lim_{x \to 0} \tan x = \tan(0) = 0$

② $\lim_{x \to \pi} (x \cos x) = \left(\lim_{x \to \pi} x\right)\left(\lim_{x \to \pi} \cos x\right) = \pi \cos(\pi) = -\pi$

③ $\lim_{x \to 0} \sin^2 x = \lim_{x \to 0} (\sin x)^2 = 0^2 = 0$

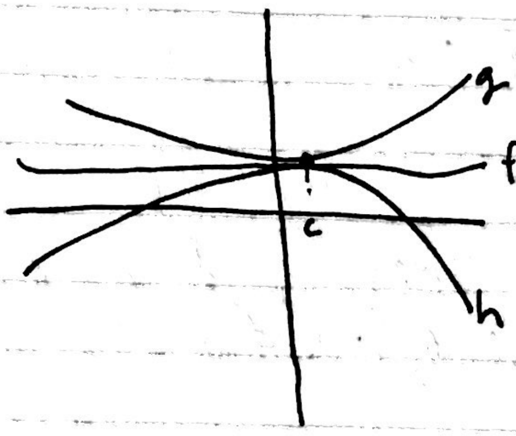
The Squeeze Theorem

The Squeeze Theorem

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$$

then $\lim_{x\to c} f(x)$ exists and is equal to L



if $\lim_{x\to c} h(x) = \lim_{x\to c} g(x) = L$

then $\lim_{x\to c} f(x) = L$

Two Special Trig Limits

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

e.g.

$$\lim_{x\to 0} \frac{\tan x}{x} = \frac{\frac{\sin x}{\cos x}}{x}$$

$$= \frac{\sin x}{\cos x} \cdot \frac{1}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x} = 1 \cdot \frac{1}{1}$$

$$= 1$$

Definition of Continuity

continuous at c if

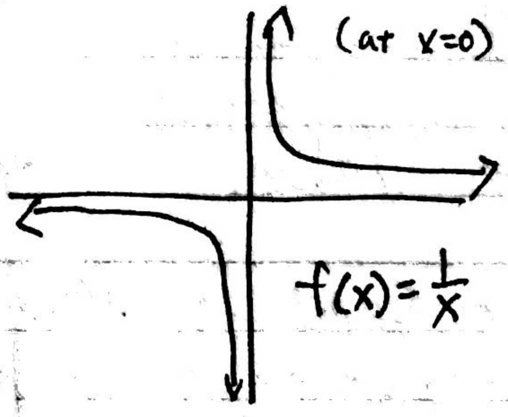
1. f(x) is defined
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

function is <u>continuous on an open interval (a, b)</u> if it is continuous at each point in the interval

function is <u>everywhere continuous</u> if it's continuous on $(-\infty, \infty)$

Nonremarable discontinuity

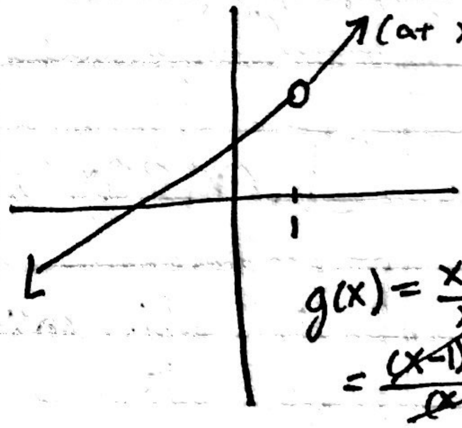
(at x=0)



$f(x) = \frac{1}{x}$

Remarable discontinuity

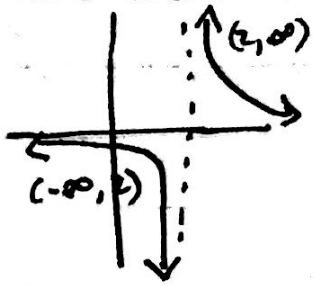
(at x=1)



$g(x) = \frac{x^2-1}{x-1}$

$= \frac{(x-1)(x+1)}{\cancel{(x-1)}}$

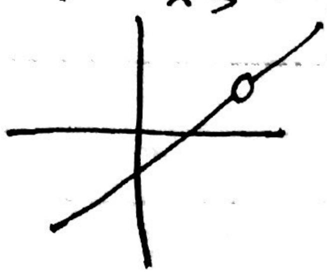
e.g. $f(x) = \frac{1}{x-2}$



$f(x) = \frac{1}{x-2}$

- not everywhere continuous
- continuous over domain
- not continuous at x=2
- non remarable

$f(x) = \frac{x^2-5x+6}{x-3}$



- everywhere continuous except at x=3
- remarable discontinuity
- not continuous at x=3
- limit exists

Existence of a Limit

limit exists only if

$\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = L$

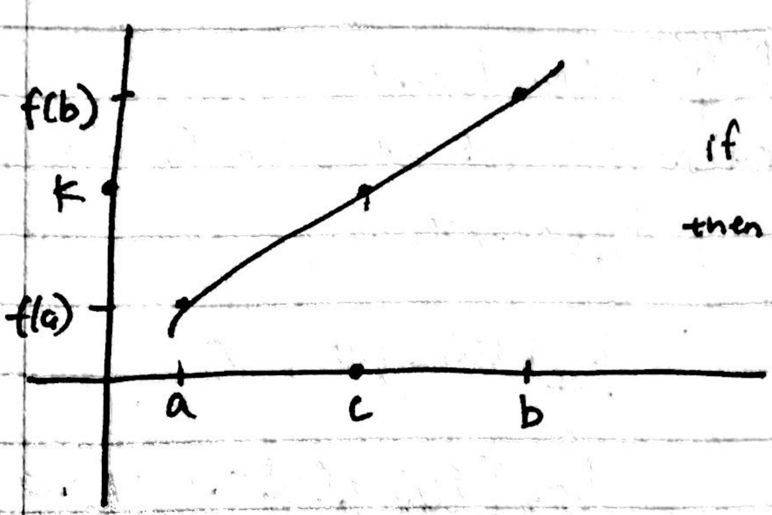
Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c

1. bf Scalar multiple
2. $f \pm g$ Sum/difference
3. fg product
4. $\frac{f}{g}$, $g(c) \neq 0$ quotient

Intermediate Value Theorem

If f is continuous on $[a,b]$, $f(a) \neq f(b)$
k is any number between $f(a)$ and $f(b)$
then there exists a number c in $[a,b]$ that $f(c) = k$

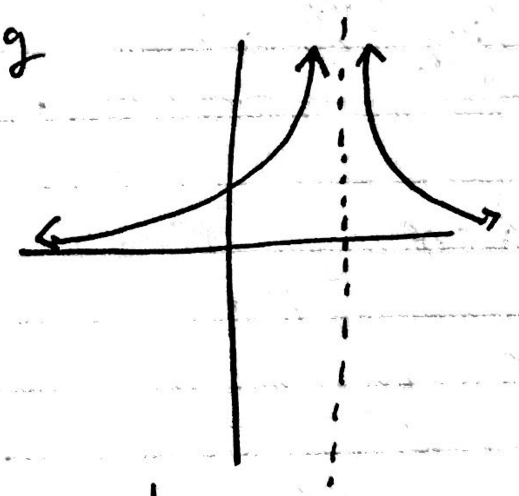


if $f(a) \le K \le f(b)$

then $f(c) = K$

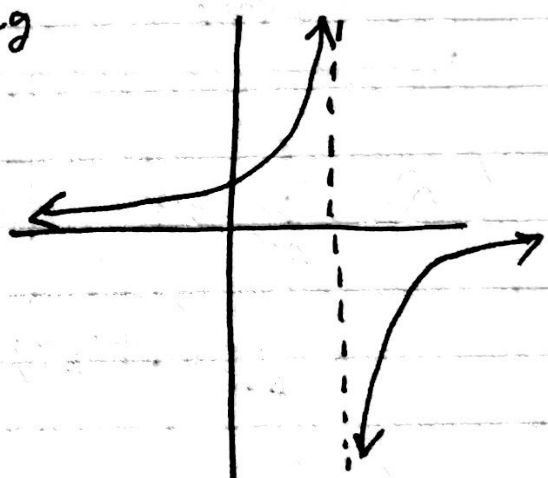
Infinite Limits

e.g



$$\lim_{x \to 1} \frac{1}{(x-1)^2} = \infty$$

eg



$$\lim_{x \to 1^-} \frac{-1}{x-1} = \infty$$

$$\lim_{x \to 1^+} \frac{-1}{x-1} = -\infty$$

$$\lim_{x \to 1} \frac{-1}{x-1} = \text{DNE}$$

Vertical Asymptote

to find VA, set denominator to 0

eg $f(x) = \frac{1}{2(x+1)}$ $\quad x = -1$ is a vertical asymptote

Properties of Infinite Limits

let c and L be real numbers, f and g be functions

$\lim_{x\to c} f(x)=\infty$ $\lim_{x\to c} g(x)=L$

1. sum/difference : $\lim_{x\to c}[f(x)\pm g(x)]=\infty$

2. Product : $\lim_{x\to c}[f(x)\,g(x)]=\infty$ $L>0$

$\lim_{x\to c}[f(x)\,g(x)]=-\infty$ $L<0$

3. Quotient : $\lim_{x\to c}\frac{g(x)}{f(x)}=0$

a) $\lim_{x\to 0} 1=1$ $\lim_{x\to 0}\frac{1}{x^2}=\infty$ $\Rightarrow$ property 1

$\lim_{x\to 0}\left(1+\frac{1}{x^2}\right)=\infty$

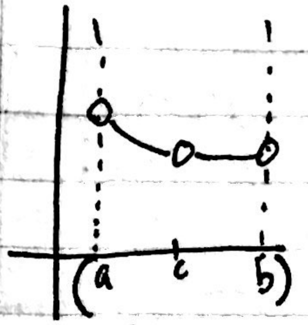
b) $\lim_{x\to 0^+} 3=3$ $\lim_{x\to 0^+}\cot x=\infty$ $\Rightarrow$ property 2

$\lim_{x\to 0^+} 3\cot x=\infty$

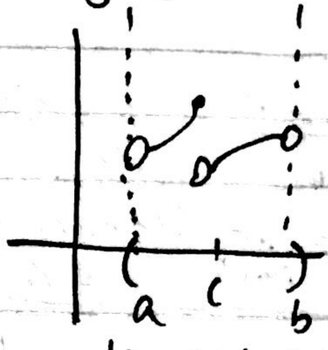
c) $\lim_{x\to 1^-}(x^2+1)=2$ $\lim_{x\to 1^-}(\cot \pi x)=-\infty$ $\Rightarrow$ property 3

$\lim_{x\to 1^-}\frac{x^2+1}{\cot \pi x}=0$

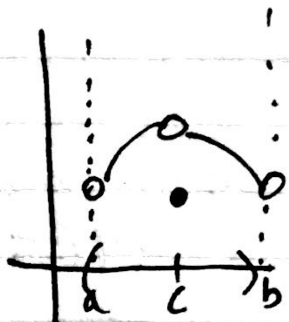
Non Continuous graphs



$f(c)$ is undefined



$\lim_{x\to c} f(x) = DNE$



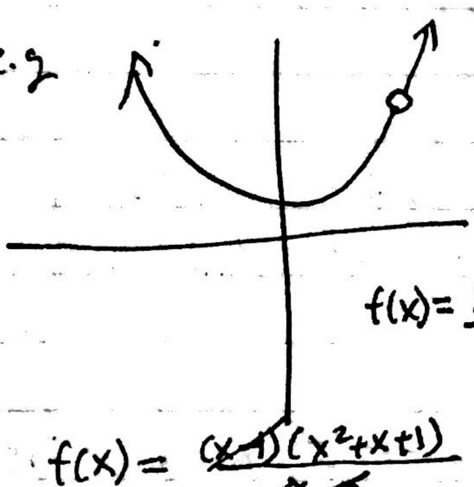
$\lim_{x\to c} f(x) \neq f(c)$

Functions that Agree at all but One Point

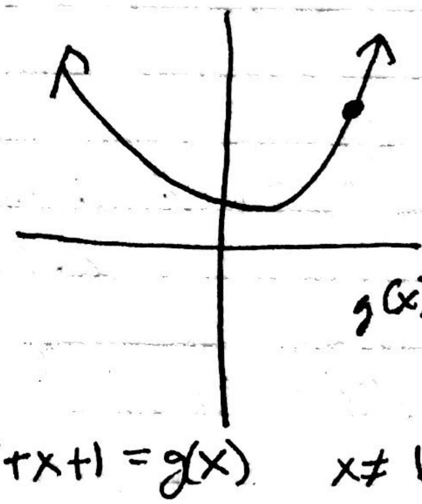
let c be a real number and $f(x) = g(x)$ for all $x \neq c$ in open interval containing c. If the limit of $g(x)$ as x approaches c exists, then limit of $f(x)$ also exists and

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x)$$

e.g



$f(x) = \dfrac{x^3-1}{x-1}$



$g(x) = x^2+x+1$

$$f(x) = \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} = x^2+x+1 = g(x) \qquad x \neq 1$$

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}}$$

$$= 1^2+1+1$$

$$= 3$$